TRANSPORT PROCESSES IN BODIES WITH A LARGE NUMBER OF CRACKS

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The problem of the effective conductivity is examined for a body with a large number of cracks or cracklike inclusions.

The case of cracks, considered as the limiting case of inclusions, is distinguished by two peculiarities. Firstly, because of the smallness of the volume fraction of the substance they contain, cracks can influence the effective properties of a medium only if the properties of the surrounding material are so radically distinct from the properties of this substance that the effect of the smallness of its volume fraction is compensated or overlapped. Hence, arbitrary approaches, operating on the volume fractions of the substances in the inclusions, are not applicable to cracks. Secondly, in contradistinction to volume inclusions, small deformations of the medium can definitely influence both the magnitude and the symmetry of the effective characteristics in the case of cracks.

1. Let us examine the effects mentioned in the small concentration approximation when the mean spacing between cracks is sufficiently large so that their interaction can be neglected. For definiteness, let us speak of the heat conductivity. In the approximation mentioned, the formula for the tensor of the effective heat conductivity λ_{ik} can be obtained in closed form (see [1], p. 69, for example, where the anisotropy and distribution function in the orientations and sizes need only be taken into account in an obvious manner to go over to the general case). Therefore,

$$\lambda_{ik} = \lambda^0_{ik} + (\lambda^1_{ij} - \lambda^0_{ij}) \int A_{jk} f(Y)\, dY, \qquad \int f(Y)\, dY = N. \tag{1}$$

Here the set of arguments of the distribution function f is denoted by Y, and N is the number of all inclusions per unit volume. The A_{ij} are determined by the relationship

$$\int s'_i\, dV = A_{ij} s_j, \tag{2}$$

where s_j is the temperature gradient far from the inclusion, s'_j is the same, within the inclusion; the integration is taken over the volume of the latter (over repeated subscripts, summing from 1 to 3). Therefore, if the solution of the problem of an isolated inclusion in a homogeneous external field is known, A_{ij} can be found from (2), and having been substituted into (1), the λ_{ik} can be found.

For a homogeneous ellipsoid in a homogeneous isotropic material with heat conductivity λ^0 and homogeneous external field s_i, the field within the ellipsoid is also homogeneous, where

$$(1 - m_i)\lambda^0 s'_i - m_i q'_i = \lambda^0 s_i, \tag{3}$$

in a local coordinate system defined by the axes of the ellipsoid of length $2a_i$ [1], where q'_i is the heat flux within the ellipsoid, and m_i are the depolarization coefficients dependent only on the a_i (there is no summation over i here). For oblate spheroids $a_1 = a_2 = a > c = a_3$

$$m_1 = m_2 = \frac{1}{2}(1 - m_3), \quad m_3 = \frac{1 + \alpha^2}{\alpha^3}(\alpha - \operatorname{arctg} \alpha), \tag{4}$$

where $\alpha = [(a/c)^2 - 1]^{1/2}$.

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2. If the ellipsoid is isotropic, then we have the following from (2) and (3):

$$A_{ij} = \frac{4}{3}\pi a_1 a_2 a_3 \frac{g_{ri} g_{rj}}{1+(\eta-1)m_r}, \quad \eta = \frac{\lambda^1}{\lambda^0}, \tag{5}$$

where λ^1 is the heat conductivity of its substance, and g_{ik} are the direction cosines of its axes. From (5), (4), and (1) we find

$$\lambda_{ik} = \lambda^0 [\delta_{ik} + \int (A\delta_{ik} + Bn_i n_k) a^3 f \sin\theta d\varphi d\theta dadc],$$
$$\int f(\varphi, \theta, a, c) \sin\theta d\varphi d\theta dadc = 4\pi N. \tag{6}$$

for the case of isotropic spheroids in an isotropic material. Here δ_{ik} is a unit tensor, the solid angle element is referred to 4π, $n_1 = \sin\theta\cos\varphi$, $n_2 = \sin\theta\sin\varphi$, $n_3 = \cos\theta$, are the components of the unit polar-axis vector

$$A = \frac{4}{3}\frac{\beta(\eta-1)}{4+(\eta-1)\pi\beta\zeta}, \tag{7}$$
$$B = \frac{2}{3}\beta(\eta-1)\left[\frac{1}{2\eta-(\eta-1)\pi\beta\zeta} - \frac{2}{4+(\eta-1)\pi\beta\zeta}\right],$$

where $\beta = c/a$; $\zeta = 4m_1/\pi\beta$; according to (4), $\zeta \to 1$ as $\beta \to 0$. The passage to cracks means that $\beta \ll 1$. In the case of high conductivity, the substance within the cracks ($\eta \gg 1$) is

$$A = -B = \frac{4}{3\pi} \text{ for } 1 \gg \frac{c}{a} \gg \frac{\lambda^0}{\lambda^1} \tag{8}$$

and $A = O(\beta\eta)$, $B = O(\beta\eta)$ for $1 \gg (\lambda^0/\lambda^1) \gg (c/a)$. In the case of low conductivity of this substance ($\eta \ll 1$)

$$A = O(\beta),\ B = -\frac{2}{3\pi} \text{ for } 1 \gg \frac{c}{a} \gg \frac{\lambda^1}{\lambda^0} \tag{9}$$

and $A = O(\beta)$, $B = O(\beta/\eta)$ for $1 \gg (\lambda^1/\lambda^0) \gg c/a$. Therefore, if the characteristics of the substance in the cracks differ so radically from the characteristics of the surrounding material that their corresponding relationship turns out to be very much less than the relative crack thickness, the values of A, B in (7) turn out to be finite and independent of the properties of the substance within the cracks. We call cracks possessing this property strongly exposed. We shall henceforth neglect the influence of cracks not strongly exposed. The influence of such cracks for which both of the small parameters being compared are of the identical order of magnitude is thereby neglected in comparison with the contribution of the strongly exposed cracks, as is possible when the fraction of the former is small.

For example, let the crack distribution be isotropic: f is independent of the angles. Then $\lambda_{ik} = \lambda\delta_{ik}$ and substituting (8) or (9) into (7), we find

$$\lambda = \lambda^0\left(1 + \frac{32}{9}v\right) \text{ for } 1 \gg \frac{c}{a} \gg \frac{\lambda^0}{\lambda^1}, \tag{10}$$

$$\lambda = \lambda^0\left(1 - \frac{8}{9}v\right) \text{ for } 1 \gg \frac{c}{a} \gg \frac{\lambda^1}{\lambda^0}. \tag{11}$$

Here $v = N\bar{a}^3$ (the bar denotes the statistical average) is a small parameter with the meaning of the volume of a perturbation domain induced by the crack in the volume of the material per crack. As a comparison, we have the following in the case of spherical inclusions of radius a [1]:

$$\lambda = \lambda^0\left(1 + 4\pi\frac{\lambda^1 - \lambda^0}{\lambda^1 + 2\lambda^0}v\right), \tag{12}$$

where $v = N\bar{a}^3$. It is seen that the influence of strongly exposed cracks is qualitatively analogous to the influence of spherical inclusions of the same radius. Quantitatively, spherical inclusions, as compared to strongly exposed cracks of the same radius, result in a rise in λ greater by $(9\pi/8) \simeq 3.5$ times in the case of their high conductivity and in a reduction in λ greater by $(9\pi/4) \simeq 7$ times in the case of low conductivity.

For cracks perpendicular to the x_3 axis, we find $\lambda_{ik} = 0$ (for $i \neq k$), $\lambda_{11} = \lambda_{22} = \lambda_t$, $\lambda_{33} = \lambda_l$ analogously, where

$$\lambda_t = \lambda^0\left(1 + \frac{16}{3}v\right), \quad \lambda_l = \lambda^0 \quad \text{for} \quad 1 \gg \frac{c}{a} \gg \frac{\lambda^0}{\lambda^1}, \tag{13}$$

$$\lambda_t = \lambda^0, \quad \lambda_l = \lambda^0\left(1 - \frac{8}{3}v\right) \quad \text{for} \quad 1 \gg \frac{c}{a} \gg \frac{\lambda^1}{\lambda^0}. \tag{14}$$

Therefore, the medium is transversally isotropic, and in the case of high conductivity the cracks influence the effective conductivity only in parallel planes, and in the case of low conductivity, only in perpendicular directions.

3. Let us assume that the material is transversally isotropic $\lambda_{11}^0 = \lambda_{22}^0 = \lambda_t^0$, $\lambda_{33}^0 = \lambda_l^0$ and the inclusions are the same as in Sec. 2 and oriented with the polar axis along the x_3 axis. As is known, the solution for this case can be obtained on the basis of the solution for an anisotropic ellipsoid in an isotropic medium by using a similarity transformation [1]:

$$\{x_1, x_2, x_3\} \to \{x_1, x_2, x_3(\lambda_t^0/\lambda_l^0)^{1/2}\}.$$

The conductivity of the material hence becomes equal to λ_t^0, the polar axis of the spheroid is $2c(\lambda_t^0/\lambda_l^0)^{1/2}$, its conductivity along it is $\lambda_l^1 = \lambda^1\lambda_t^0/\lambda_t^0$ while in the transverse direction $\lambda_t^1 = \lambda^1$. Hence, we find the following from (1), (2), (3):

$$\lambda_{t,l} = \lambda_{t,l}^0[1 + \int a^3 A_{t,l} f(a, c)\,da\,dc], \qquad \int f\,da\,dc = N,$$

$$A_t = \frac{4}{3}\pi\frac{\beta(\eta^* - 1)}{1 + (\eta^* - 1)m_1^*}, \quad A_l = \frac{4}{3}\pi\frac{\beta[\eta^*(\lambda_t^0/\lambda_l^0) - 1]}{1 + [\eta^*(\lambda_t^0/\lambda_l^0) - 1]m_3^*}, \tag{15}$$

where $\lambda_{11} = \lambda_{22} = \lambda_t$, $\lambda_{33} = \lambda_l$, $\eta^* = (\lambda^1/\lambda_t^0)$, $\beta = (c/a)$, m_i^* are expressed by (4) with α replaced by $\alpha^* = [(\lambda_l^0/\beta^2\lambda_t^0) - 1]^{1/2}$. In particular, it hence follows that

$$\lambda_t = \lambda_t^0\left(1 + \frac{16}{3}\sqrt{\frac{\lambda_l^0}{\lambda_t^0}}\,v\right), \quad \lambda_l = \lambda_l^0 \tag{16}$$

for

$$(\lambda_l^0/\lambda_t^0)^{1/2} \gg \beta \gg (\lambda_l^0\lambda_t^0)^{1/2}/\lambda^1. \tag{17}$$

The right-hand inequality denotes that the cracks are strongly exposed, while the left (when λ_t^0 and λ_l^0 are of the same order) is the ordinary condition of smallness of the relative crack thickness.

4. Let us examine the influence of mechanical stresses. Let β_0 be the relative half-thickness of a crack in an unstressed body. Considering that the substance within the cracks does not exert any mechanical resistance, the additional half-exposure can be calculated, because of the smallness of β_0, by using (10.129) from [2] for an infinitely thin slit. Then we obtain, analogously to [3],

$$\beta = \beta_0 + \frac{4(1 - \nu_0^2)}{\pi E_0}\sigma_{ik}n_i n_k, \tag{18}$$

where σ_{ik} is the applied stress tensor. Now, β from (18) should be introduced under the condition of strong exposure of the cracks. Thus, for isotropically distributed cracks, we have in place of condition (8)

$$\sigma_{ik}n_i n_k \gg \frac{\pi E_0}{4(1 - \nu_0^2)}\left(\frac{\lambda^0}{\lambda^1} - \beta_0\right). \tag{19}$$

If $(\lambda^0/\lambda^1) - \beta_0 > 0$, then the cracks in the unstressed body are not strongly exposed, and in order to make their exposure strong it is necessary, according to (19), to apply tensile stresses. In the opposite case, the cracks in the unstressed body are strongly exposed and, according to (19), can become not strongly exposed by applying compressive stresses. For example, let the loading be uniaxial so that $\sigma_{ik}n_i n_k = \sigma\cos^2\theta$, where θ is the angle between the loading axis and the normal to the crack. Then λ_{ik} is found by means of (6) and (8), with f independent of the angles, where the integration is over all φ and over θ in the interval $(\pi/2) - \psi_* < \theta < (\pi/2) + \psi_*$, where $\sin^2\psi_* \equiv h_*^2 \simeq [(\lambda^0/\lambda^1) - \beta_0]\pi E_0/4(1 - \nu_0^2)\sigma$ if the initial crack exposure is strong, and with the interval $(\pi/2) - \psi_{**} < \theta < (\pi/2) + \psi_{**}$ discarded, where $\sin^2\psi_{**} \equiv h_{**}^2 \simeq [\beta_0 - (\lambda^0/\lambda^1)]\pi E_0/4(1 - \nu_0^2)(-\sigma)$ if it is not strong. Here the stress σ is assumed to be adequate so that $h_* < 1$ (for $\sigma > 0$) or $h_{**} < 1$ (for $\sigma < 0$); in the opposite case its influence is negligible. Integrating and using the notation $\lambda_{11} = \lambda_{22} = \lambda_t$, $\lambda_{33} = \lambda_l$ (the medium effectively becomes transversally isotropic), for the case of the initial crack exposure not strong we find

$$\lambda_t = \lambda^0 \left[1 + \frac{8}{9}(4 - 3h_* - h_*^3)v\right],$$
$$\lambda_l = \lambda^0 \left[1 + \frac{16}{9}(2 - 3h_* + h_*^3)v\right], \quad h_* \simeq \left\{\frac{[(\lambda^0/\lambda^1) - \beta_0]\pi E_0}{4(1 - \nu_0^2)\sigma}\right\}^{1/2}, \tag{20}$$

and for the case of a strong initial exposure,

$$\lambda_t = \lambda^0 \left[1 + \frac{8}{9}(3h_{**} + h_{**}^3)v\right],$$
$$\lambda_l = \lambda^0 \left[1 + \frac{16}{9}(3h_{**} - h_{**}^3)v\right], \quad h_{**} \simeq \left\{\frac{[\beta_0 - (\lambda^0/\lambda^1)]\pi E_0}{4(1 - \nu_0^2)(-\sigma)}\right\}^{1/2}. \tag{21}$$

5. All the preceding formulas are valid even if a small concentration of inclusions is introduced into the medium containing inclusions of much smaller size. Relative to the inserted large inclusions, such a medium can be considered homogeneous and possessing an effective conductivity. Let us assume that there is a broad size distribution of inclusions, and inclusions of similar size to which an infinitely small concentration corresponds are distributed uniformly in the space between the large-size inclusions; the total inclusion concentration can hence be finite. Then the resultant effective conductivity can be found by a step-by-step application of the low-concentration approximation. Thus, in the case of spherical inclusions we must replace λ by $\lambda + d\lambda$, λ^0 by λ, v by dv in (12) to determine the increment in the heat conductivity $d\lambda$ as a result of adding an infinitesimal fraction of inclusions whose size considerably exceeds the size of all those already introduced, and then integrating we find

$$v = \frac{1}{4\pi} \ln\left[\left(\frac{\lambda^1 - \lambda^0}{\lambda^1 - \lambda}\right)^3 \frac{\lambda}{\lambda^0}\right]. \tag{22}$$

In the case of quite high $(\lambda^1/\lambda^0) \gg 1$ or quite low $(\lambda^1/\lambda^0) \ll 1$ conductivity of the inclusions, we find

$$\lambda = \lambda^0 \exp(\varkappa v), \tag{23}$$

where $\varkappa = 4\pi$ in the former case and $\varkappa = -2\pi$, in the latter. We also have the same formula with $\varkappa = (32/9)$ and $\varkappa = -(8/9)$ in the same cases for cracks [see (11) and (12)].

The medium will be transversally isotropic in each step for a system of parallel cracks, so that by applying an analogous procedure to (17), we find for the case of high conductivity

$$\lambda_t = \lambda^0 \left(1 + \frac{8}{3}v\right)^2, \quad \lambda_l = \lambda^0. \tag{24}$$

The low-conductivity case can be examined in the same way. In the case of cracks $v = N\bar{a}^3 = \Sigma N_i a_{(i)}^3$ in these formulas, where N_i is the number of cracks of radius $a_{(i)}$ per unit volume of the medium, $N = \Sigma N_i$. For spherical (and generally volume) inclusions, v is analogously expressed; however, N_i are the numbers of inclusions per unit volume of the medium in the space between all the inclusions of radius greater than $a_{(i)}$, since it is necessary to take into account the intrinsic volume of the inclusions. In this latter case, evidently $\Delta v_i = N_i a_{(i)}^3 = (3/4\pi)\Delta w_i/(1 - w_i)$, where w_i and Δw_i are the volume fractions of inclusions of radii greater than $a_{(i)}$ and of radius $a_{(i)}$, respectively. Going over to the asymptotic of the broad size distribution and here replacing the increment by the differential, after integration we find

$$v = -\frac{3}{4\pi} \ln(1 - w). \tag{25}$$

Considering the volume fraction of inclusions as argument from the very beginning, Bruggeman [4] established a power-law dependence obtained from (22), (25) by eliminating v, without indicating, however, that a broad size distribution is required for its validity. The value of this requirement for the validity of the Bruggeman results (and the fact that it is not valid for inclusions of mutually similar size) was apparently first conceived in [5]. In connection with the problem of the effective elastic characteristics of bodies with a large number of cracks, the asymptotic of a broad size distribution was considered in [6].

The value of the results obtained by using the differential procedure mentioned emerges beyond the framework of the approximation of a broad size distribution, however. For inclusions whose properties differ strongly from the properties of the surrounding material, including cracks, these results are directly applicable independently of the size distribution, including the case of completely identical sizes, in a wide range of variation of the concentration under the condition that the inclusions remain isolated, i.e., the formation of pairs, triplets, etc., can be neglected (in the opposite case they should be taken into account as new objects), and that a spatial arrangement of inclusions compatible with this requirement and the conservation of this effective symmetry of the medium be sufficiently random. However, it should be emphasized that such a universality of the result holds for volume inclusions only when v is taken as the argument. The dependence on the size distribution is manifest in the expression of the result in terms of the volume fraction of inclusions. Thus, we have the dependence (25) in the case of a very broad distribution and $v = (3/4\pi)w$, in the case of identical dimensions. The argument is only v for cracks and similar questions do not arise.

The possibility of the considered broadening of the range of applicability of the results obtained on the basis of the mentioned differential procedure is shown and confirmed experimentally for kindred problems about the effective elastic characteristics in our joint research with A. S. Vavakin.

NOTATION

λ_{ik}^{s}	is the heat conductivity tensor (s = 0, material; s = 1, inclusions or cracks; no superscript, effective);
λ^{s}	is the isotropic heat conductivity;
λ_{t}^{s}, λ_{l}^{s}	are the heat conductivity in the case of transversal isotropy in the isotropy plane and in the perpendicular direction, respectively;
a_i	is the half-length of the axes of an ellipsoidal inclusion;
a	is the radius of a spheroidal inclusion at the equator (particularly of a crack);
c	is the half-length of its polar axis;
n_i	is the unit vector of this axis;
φ, θ	are the longitude and latitude defining its direction;
Y	is the set of orientation and size parameters of an ellipsoidal inclusion;
f(Y)	is the distribution function of such inclusions;
N	is the number of all cracks or inclusions per unit volume $v = N\bar{a}^3$;
$\bar{a}^3$	is the mean cubic radius of a crack or an inclusion equator;
w	is the volume fraction of spherical inclusions;
$\beta = c/a$;	
β_0	is the value of β in an unstressed body;
$\eta = \lambda^1/\lambda^0$;	
$\eta^* = \lambda^1/\lambda_t^0$;	
E_0, ν_0	are the Young's modulus and Poisson's ratio of the material;
σ_{ik}	is the mechanical stress tensor;
σ	is the uniaxial stress, tensile ($\sigma > 0$) or compressive ($\sigma < 0$).

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